Use hats for estimators

$E(S^2)=\sigma^2$

$E(S)=\sigma$

$Cov(aX,bY)=ab\,Cov(X,Y)$
$Cov(X+a,Y+b)=Cov(X,Y)$
$Cov(X,X)=Var(X)$
$Cov(X,Y)=E(XY)-E(X)E(Y)$

$Cov(X+Y,W+Z)=Cov(X,W)+Cov(Y,W)+Cov(X,Z)+Cov(Y,Z)$
$Var(X+Y)=Var(X)+Var(Y)+2Cov(X,Y)$
$Corr = Cov/\sigma_X\sigma_Y \quad -1\le Corr\le 1$
$-\sigma_X\sigma_Y \le Cov \le \sigma_X\sigma_Y$

If $X_1,\dots,X_n$ i.i.d. norm r.v.s then
a. $\bar{x}$ and s are indep
b. $(n-1)S^2/\sigma^2 \sim \chi^2_{\nu=n-1}$

$Var(S^2)=2\sigma^4/n-1$

$\int u\,dv = uv-\int v\,du$
$\int uv\,dx = u\int v\,dx-\int u'(\int v\,dx)dx$

640:481 Austin DeCicco Final Sheet

$\phi_X = E(e^{tX})=\int e^{tx}f(x)dx$
$E(X^r)=\frac{d^r}{dt^r}\phi_X(t=0)$
$\mu_r'=E(X^r)$ $\mu_r=E((X-\mu_1')^r)$

Gamma: $f(x)=\frac{1}{\beta^\alpha\Gamma(\alpha)}x^{\alpha-1}e^{-x/\beta}$ $\phi_X=1/(1-\beta t)^\alpha$ | $\Gamma(\alpha)=\int_0^\infty y^{\alpha-1}e^{-y}dy=(\alpha-1)\Gamma(\alpha-1)=(\alpha-1)!$

$\alpha,\beta,x>0$ $E(X)=\alpha\beta$ $Var(X)=\alpha\beta^2$ | $\Gamma(1/2)=\sqrt{\pi}$ $e^x=\sum_{k=0}^\infty x^k/k! = \lim_{n\to\infty}(1+x/n)^n$ | $\chi^2_{\alpha,\nu}$

Exp: $f(x)=\frac{1}{\lambda}e^{-x/\lambda}$ $\phi_X=\frac{1}{1-\lambda t}$ if $t-\frac{1}{\lambda}<0$ | Chi: $f(x)=\frac{1}{2^{\nu/2}\Gamma(\nu/2)}x^{\frac{\nu-2}{2}}e^{-x/2}$ $\phi_X=(1-2t)^{-\nu/2}$

$x,\lambda>0$ $E(X)=\lambda$ $Var(X)=\lambda^2$ | Special case of gamma $\alpha=\nu/2$ $\beta=2$ | $x,\nu>0$ $E(X)=\nu$ $Var(X)=2\nu$ $\sum_{i=1}^n Z_i^2=\chi^2_{\nu=n}$ for Z_i i.i.d.

Norm: $f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\cdot e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$ $\phi_X=e^{t\mu+\frac{1}{2}t^2\sigma^2}$ | Geo: $f(x)=p(1-p)^{x-1}$ $\phi_X=pe^t/1-(1-p)e^t$ $E(X)=1/p$ $V(X)=\frac{1-p}{p^2}$

Bin: $f(x)=\binom{n}{x}p^x(1-p)^{n-x}$ $\phi_X=(pe^t+1-p)^n$ $E(X)=np$ $Var(X)=np(1-p)$ | Median: $n=2m+1$ $\tilde{x}=Y_{m+1}$

Markov: $P(X\ge a)\le\frac{E(X)}{a}$ $x\ge0$ | Chebyshev: $P(|X-\mu_X|\ge k)\le\sigma_X^2/k^2$ $k>0$ | $h(\tilde{x})=\frac{(2m+1)!}{m!m!}\left[\int_{-\infty}^{\tilde{x}}f(x)dx\right]^m f(\tilde{x})\left[\int_{\tilde{x}}^{\infty}f(x)dx\right]^m$

Order: $g_r(y_r)=\frac{n!}{(r-1)!(n-r)!}\left[\int_{-\infty}^{y_r}f(x)dx\right]^{r-1}f(y_r)\left[\int_{y_r}^{\infty}f(x)dx\right]^{n-r}$ $n=2m$ $\tilde{x}=\frac{1}{2}(Y_m+Y_{m+1})$

$\mu_2=Var(X)$

If $Y=aX+b$
$\phi_Y=e^{tb}M_X(at)$

If X & Y indep
$Z=X+Y$ $\phi_Z=\phi_Y\phi_X$

$\bar{x}=\sum x_i/n$ $s^2=\frac{\sum(x_i-\bar{x})^2}{n-1}$ $E(\bar{x})=\mu$ $Var(\bar{x})=\sigma^2/n$ $P(\mu-c\le\bar{x}\le\mu+c)\ge1-\frac{\sigma^2}{nc^2}$ $Z=\frac{\bar{x}-\mu}{\sigma/\sqrt{n}}$ $\lim_{n\to\infty}\phi_Z=e^{\frac{1}{2}t^2}$

Methods of moments: r-th sample moment about origin, $M_r'=\sum x_i^r/n$ $M_1'=\bar{x}$ $M_2'=\frac{x_1^2+\dots+x_n^2}{n}$
by SL of LN, $M_r'\to E(X^r)$, thus set $M_r'=\mu_r'$ for estimators.

MLE: $L(\theta)=f(x_1,x_2,\dots,x_n|\theta)$ Take $d/d\theta$ and set to 0. $\ln(L(\theta))$ might be easier

SL: $\lim_{n\to\infty}\sum x/n=\mu$ with prob 1 | CLT: $\lim_{n\to\infty}P\left(\frac{\sum(x_i-\mu)}{\sqrt{\sum\sigma_i^2}}\le a\right)=\Phi(a)$ | WL: $P(|\bar{x}-\mu|\ge\varepsilon)\to0$ as $n\to\infty$

	$H_0=T$	$H_0=F$
R	1	X
A	X	2

$P(1)=\alpha$
$P(2)=\beta$
Power: $1-\beta$

Unbiased if $E(\hat{\theta})=\theta$ | Asymptotically unbiased if $\lim_{n\to\infty}E(\hat{\theta})=\theta$ | CRLB: $\frac{1}{nE[(\ln(f(x))^d/d\theta)^2]}$ or $\frac{-1}{nE[\frac{d^2}{d\theta^2}\ln(f(x))]}$

$CRLB(\hat{\mu})=\sigma^2/n$ | $CRLB(\hat{\sigma}^2)=2\sigma^4/n$ | Critical Region = Rejection Region | t-score: $t_{\alpha,\nu}$ | f-score: f_{α,ν_1,ν_2}

$T=\frac{Z}{\sqrt{Y/\nu}}$ for $Y\sim\chi^2_\nu$ | $f(t)=\left(\Gamma(\frac{\nu+1}{2})/\sqrt{\pi\nu}\,\Gamma(\frac{\nu}{2})\right)\cdot(1+\frac{t^2}{\nu})^{-\frac{\nu+1}{2}}$ $T=\frac{\bar{x}-\mu}{s/\sqrt{n}}\sim t_{n-1}$ | $f_{\alpha,\nu_1,\nu_2}=\frac{1}{f_{1-\alpha,\nu_2,\nu_1}}$

$F=\frac{U/\nu_1}{V/\nu_2}=(\nu_2/\nu_1)(U/V)$ for $U\sim\chi^2_{\nu_1}$, $V\sim\chi^2_{\nu_2}$ | $g(f)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\left(\frac{\nu_1}{\nu_2}\right)^{\nu_1/2}f^{(\nu_1/2)-1}\left(1+\frac{\nu_1}{\nu_2}f\right)^{-\frac{1}{2}(\nu_1+\nu_2)}$, $f>0$

For $F\sim f_{\nu_1,\nu_2}$, $1/F\sim f_{\nu_2,\nu_1}$

$F=\sigma_2^2s_1^2/\sigma_1^2s_2^2\sim f_{n_1-1,n_2-1}$ | $H_0:\mu=4$ simple, $H_1:\mu<4$ composite

Difference of means: indep $Z=\frac{(\bar{x}_1-\bar{x}_2)-(\mu_1-\mu_2)}{\sqrt{\sigma_1^2/n_1+\sigma_2^2/n_2}}$, $S_p^2=\frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}$, $T=\frac{(\bar{x}_1-\bar{x}_2)-(\mu_1-\mu_2)}{S_p\sqrt{1/n_1+1/n_2}}\sim t_{n_1+n_2-2}$

Proportions: $Z=\frac{(\sum X_i)-np}{\sqrt{np(1-p)}}$ $Z=\frac{\bar{x}-p}{\sqrt{p(1-p)/n}}$ | Difference of proportions: indep $Var(\bar{x}_i)=\frac{p_i(1-p_i)}{n_i}$ $Z=\frac{(\bar{x}_1-\bar{x}_2)-(p_1-p_2)}{\sqrt{Var(x_1)+Var(x_2)}}$

Estimate σ^2 with $\frac{(n-1)s^2}{\sigma^2}\sim\chi^2_{\nu=n-1}$ | Estimate σ_1/σ_2 with $\sigma_2^2s_1^2/\sigma_1^2s_2^2\sim f_{n_1-1,n_2-1}$

$$J = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} & \cdots \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} & \cdots \end{vmatrix}$$

Neymann-Pearson: $H_0 = \theta_0$ and $H_1 = \theta_1$, $L_0 = \prod f(x_i|\theta_0)$, $L_1 = \prod f(x_i|\theta_1)$ $K > 0$

Critical Region C of size α, C is MPCR if $L_0/L_1 \le K$ inside C, $L_0/L_1 \ge K$ outside C

Notes: Rename K and to determine test stat then calc K with a known pdf or CI

Likelihood Ratio: $\omega \cap \omega^c = \emptyset$, $\omega \cup \omega^c = \Omega$ LRT makes no claims about power

$L_\omega(x_1,\dots,x_n) = \max \prod f(x_i|\theta)$ for $\theta \in \omega$ $\lambda = L_\omega / L_\Omega$ Note: For L_Ω use $\hat{\theta}$

$L_\Omega(x_1,\dots,x_n) = \max \prod f(x_i|\theta)$ for $\theta \in \Omega$ $0 \le \lambda \le 1$, $0 < K < 1$

When $H_0 = T$ we expect $\lambda \approx 1$ and when $H_0 = F$ we expect $\lambda \approx 0$.

Thus $\lambda \le K$ defines the critical region.

P-value: Observed level of significance, lowest α at which H_0 is rejected

Regression: $\mu_{Y|X} = \int_y y \cdot w(y|x)\,dy$ $w(y|x) = f(x,y)/f(x)$ $f(x) = \int_y f(x,y)\,dy$

$\mu_{Y|X} = \alpha + \beta X$ $\rho = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \sigma_{XY}/\sigma_X\sigma_Y$ $\beta = \sigma_{XY}/\sigma_X^2$ $\alpha = \mu_Y - \beta\mu_X$

$\mu_{Y|X} = \mu_Y + \beta(x - \mu_X)$ $\mu_{X|Y} = \mu_X + \rho(\sigma_X/\sigma_Y)(y - \mu_Y)$ Assume $Var(Y|X) = Var(Y) = \sigma_Y^2$

X indep var, Y dependent var If Y norm: $w(y|x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-(y-(\alpha+\beta x))^2/2\sigma^2}$

$S_{xx} = \sum (x_i - \bar{x})^2 = \overrightarrow{(x_i - \bar{x})} \cdot \overrightarrow{(x_i - \bar{x})}$ $S_{xy} = \sum (x_i - \bar{x})(y_i - \bar{y}) = \overrightarrow{(x_i - \bar{x})} \cdot \overrightarrow{(y_i - \bar{y})}$

$\hat{\beta} = S_{xy}/S_{xx}$ $\hat{\alpha} = \bar{y} - \hat{\beta}\bar{x}$ $E(\hat{\beta}) = \beta$ $Var(\hat{\beta}) = \sigma^2/S_{xx}$

$\hat{\sigma}^2 = (S_{yy} - \hat{\beta} S_{xy})/n$

Normal Equations for α, β, σ

$\sum (y_i - \alpha - \beta x_i) = 0$

$\sum (y_i - \alpha - \beta x_i) x_i = 0$

$n - \sum (y_i - \alpha - \beta x_i)^2/\sigma^2 = 0$

$n\hat{\sigma}^2/\sigma^2 \sim \chi^2_{\nu = n-2}$ and indep of $\hat{\beta}$

$\frac{\hat{\beta} - \beta}{\hat{\sigma}} \sqrt{\frac{(n-2)S_{xx}}{n}} \sim t_{n-2}$

$\frac{\hat{\beta} - \beta}{\sigma/\sqrt{S_{xx}}} \sim Z$

Transform One Var: Discrete $Y = u(x)$ is one to one, $u^{-1}(Y) = X$ then $f_Y = f_X(u^{-1}(y))$ and change $\boxed{\text{domain}}$

Otherwise for diffable & monotone $u(x)$ for $f(x) \ne 0$

$f_Y = f_X(u^{-1}(y)) \cdot \left|\frac{d}{dy} u^{-1}(y)\right|$ provided $u'(x) \ne 0$ else $g(y) = 0$

Transform ≥ 2 Var: $f(x_1, x_2)$ given and $Y = u(x_1, x_2)$

Sub expressions for $x_i = \text{func}(Y, x_j)$ obtained from same expression of x_i

Discrete: $g(y, x_2) = f(x_1, x_2)\left|\frac{\partial x_1}{\partial y}\right|$, $g(x_1, y) = f(x_1, x_2)\left|\frac{\partial x_2}{\partial y}\right|$

then integrate out x_j to get f_Y. $\boxed{\text{Bounds}}$

Otherwise: $Y_1 = u_1(x_1, x_2)$, $Y_2 = u_2(x_1, x_2)$ for partial diffable one to one

$x_1 = w_1(y_1, y_2)$, $x_2 = w_2(y_1, y_2)$

$g(y_1, y_2) = f_{x_1,x_2}(w_1(y_1,y_2), w_2(y_1,y_2)) \cdot |J|$ then integrate for f_{Y_1}.